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Research Statement

My research interests are in low-dimensional topology, in particular the topology of three- and four-dimensional manifolds. Most of the projects I have pursued follow the theme of determining the structure of a 3- or 4-manifold from information contained in its two-dimensional submanifolds.

I briefly outline my research relating to surfaces in 3-manifolds and 4-manifolds in Sections 1 and 2, respectively. In Section 3 I describe plans for developing a software package focused on visualizing curves on a surface that I believe would be of value to the mathematical community. I also discuss ways in which undergraduates in math or computer science could contribute to this project.

1. Surfaces in 3-manifolds : the simple loop conjecture

One of the most ubiquitous themes in modern topology is the connection between the topological properties of a space and its algebraic invariants. This theme manifests in three-dimensional topology when considering surfaces in 3-manifolds: an embedded surface in a 3-manifold can be **incompressible**, meaning it cannot be cut into a simpler surface inside the 3-manifold, and it can be **π_1-injective**, meaning the fundamental group of the surface embeds in the fundamental group of the 3-manifold.

It is straightforward to show that a π_1-injective surface in a 3-manifold is always incompressible. Moreover, it follows from the Loop Theorem of Papakyriakopoulos that these two notions are equivalent for certain embedded surfaces in 3-manifolds. (The surfaces in question must satisfy a technical condition called **2-sidedness**.) This result points to one of many ways in which the topology of a 3-manifold is strongly determined by its fundamental group.

Let's relax the condition that the surfaces are embedded and instead consider *immersed* surfaces in 3-manifolds. There are natural generalizations of the notions of incompressible and π_1-injective to immersed surfaces, and, as in the embedded case, π_1-injective implies incompressible. However, we can no longer appeal to the Loop Theorem to prove the converse. We arrive at:

Conjecture (Simple Loop Conjecture for 3-Manifolds)**.** *Every incompressible (2-sided) immersed closed surface in a closed 3-manifold is π_1-injective.*

The conjecture is known to hold for surfaces in certain classes of 3-manifolds ([4, 5]). In [9] I prove that the conjecture holds when the 3-manifold admits a particular geometric structure (*Sol*-geometry). However, the general case of

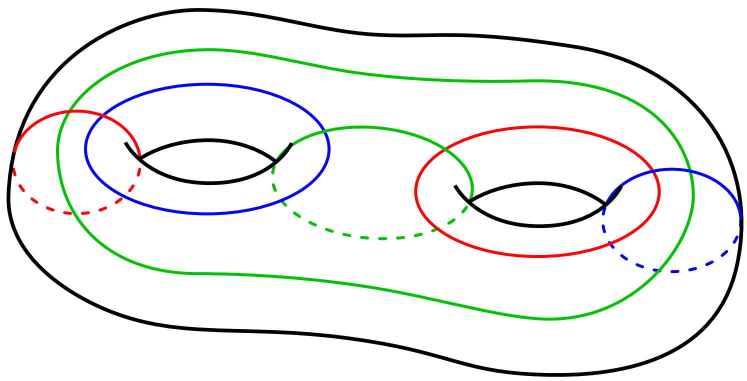

FIGURE 1. A trisection diagram for a trisection of $S^2 \times S^2$. The curves give instructions for attaching three three-dimensional handlebodies to the surface, which in turn determine how the four-dimensional pieces of the trisection fit together.

the conjecture remains open and worthy of study. Proofs of the conjecture for other classes of 3-manifolds (such as hyperbolic 3-manifolds) would further strengthen the bridge between the topology of a 3-manifold and the group theory of its fundamental group.

2. SURFACES IN 4-MANIFOLDS : TRISECTIONS

In [3], Gay and Kirby introduce the notion of a **trisection** of a 4-manifold, which is a decomposition of a smooth 4-manifold into three simple four-dimensional pieces. The three pieces intersect along a **central surface** inside the 4-manifold, and it turns out that the entire smooth structure of the 4-manifold is determined by the way in which the pieces of the trisection are attached to the surface. In fact, one may compress the information contained in a trisection into a collection of colored curves on the central surface (called a **trisection diagram**, see Figure 1), which provides a means to study the 4-manifold in question from a two-dimensional viewpoint.

It's worth mentioning that a trisection of a 4-manifold is analogous to a **Heegaard splitting** of a 3-manifold, which is a decomposition into two simple three-dimensional pieces that meet along a surface. In particular, Heegaard splittings allow one to study a 3-manifold via **Heegaard diagrams** on that surface.

My work with trisections has focused extracting information about trisected 4-manifolds from the combinatorial data in their trisection diagrams. The following are three perspectives from which I have considered this idea, along with some relevant questions for further study.

2.1. **Homology and other algebraic invariants.** In [2] we give explicit formulas that establish the following.

Theorem. *The homology and intersection form of a trisected 4-manifold is determined by the homological information in any trisection diagram for it.*

Question. *What other algebraic invariants of a 4-manifolds can we compute using trisection diagrams?*

2.2. **Curve complexes.** A trisection diagram gives rise to three **handlebody sets** inside the curve complex of the central surface. The interactions between these subcomplexes reflect the structure of the trisection. For example, one may define a notion of complexity for trisections by considering the shortest circuit contained in the union of the three handlebody sets that travels through each one.

Question. *What can be said about 4-manifolds with trisections of high complexity?*

Question. *For a fixed trisection, what does the union of the three handlebody sets in the curve complex look like?*

2.3. **Mapping class groups.** The information contained in a trisection can be expressed via **gluing maps**, which can be thought of as elements of the mapping class group of the central surface of a trisection that give instructions for gluing together the pieces of the trisection. In particular, in [10] I show that there is a correspondence between equivalence classes of trisections and certain types of double cosets in the mapping class group of the central surface.

Question. *Can we use the group-theoretic perspective provided by the mapping class group to gain insights on trisected 4-manifolds?*

3. Surface Visualization : *NormalC*

Almost all of the work I've done over the past few years has involved drawing curves on a surface. This is relatively easy to do on a chalkboard when the curves are few and of low complexity, but drawing multiple complicated curves or performing operations on curves (eg. band sums or Dehn twists) is time consuming and messy.

Thus I have often found myself wishing for a program that would let me draw and "play" with curves on a surface: ideally I could trace the curves with a mouse and then have a computer carry out operations and calculations. There currently exist some tools ([8] and [1], for example) for drawing surfaces and curves on surfaces and performing some basic operations, but at this point no program exists that provides the flexibility and ease of use that would allow someone to play with curves on a surface.

A short time ago I started developing my own vision for a software like this. The program I plan on writing, called *NormalC* ("normalcy"), will allow the user to manipulate a surface in 3-dimensions and draw curves on it using their mouse. The underlying mathematics of the program will be based on *normal curve theory*, wherein an embedded curve on a surface is determined by its pattern of intersection with the edges of a triangulation of the surface. The set of normal curves on a surface (those with a well-behaved intersection pattern) can be equated with the set of solutions to a particular system of $\mathbb{Z}$-linear equations, leading to applications in algorithmic topology. Indeed, the problem of algorithmically carrying out some curve operations has been studied [6, 7].

My undergraduate major (Computational Mathematics) focused on both math and computer science, and in college I spent a significant amount of time working on programming projects for classes I was taking and for my own leisure. Thus I am excited to work on project in computational topology, which lies in the center of my skill set. Once I finish programming the basic functions of *NormalC*, my plan is to make the software available online, include a way for other programmers to add their own modules, and then collect modules written by the community in an online database. My hope is that this program will be a useful tool for low-dimensional topologists like myself to use for experimentation and illustration.

3.1. **Undergraduate Projects.** Certain aspects of the *NormalC* project make it fitting for participation by undergraduate students who are interested in math. In particular, the connection between the linear algebra and geometry involved in normal curve theory is straightforward to explain and illustrate, and it provides an accessible starting point to a student who is unfamiliar with topology. On the other hand, normal curve theory has connections to the numerous facets of surface topology, so there are many directions that an interested student could explore.

I have several *NormalC*-related projects in mind that would be accessible and engaging to students accross a broad range of mathematical maturity. Here are a few of these ideas, in order of increasing difficulty and length.

- Learn about normal curve theory and write up a report that explains the linear algebra and topology behind it.
- Design an algorithm using normal curves to carry out a simple topological process, such as finding intersections between curves, and implement it in *NormalC*.
- Read and understand the current literature on normal curve algorithms for more complicated curve operations and implement one of them. (The result of this work could be turned into a module to be added to the online database mentioned earlier.)

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